Taking the approximate relation $2\Gamma/(\epsilon_s-\epsilon_a)_{\max}\approx 0.1$ (see Fig. 4).

$$P_0=e^{-0.1e\pi\Gamma T}=e^{-0.85\Gamma T}.$$

The mean life for decay is given by the relation $0.85\Gamma T=1$ or $\Gamma T=1.2$, which again agrees with the uncertainty principle $\Delta E\Delta T\approx 1$.

Thus, two apparently quite different approaches agree. This tends to reinforce the conclusion that adiabatic potential curves are not important in the theory of fast atomic collisions. It would be interesting to find out if a more refined collision theory would bear out these arguments.

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Calculation of the Scattering Constant from the Theory of Multiple Scattering*

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The expressions for the mean spatial and projected angles of multiple scattering are obtained using the distribution function for multiple scattering derived by Nigam, Sundaresan and Wu, and compared with those of Molière. It is shown that Molière's calculations involve the approximation of $\chi_c\sqrt{B}\rightarrow 0$. The distribution function of Nigam *et al.* is found to give correction terms which are proportional to powers of $\chi_c\sqrt{B}$ and $\chi_c\sqrt{B}\ln(\pi/\chi_c\sqrt{B})$.

I. INTRODUCTION

The theory of multiple scattering of a charged particle passing through matter has been worked out by Williams,[1] Goudsmit and Saunderson,[2] Molière,[3] Snyder and Scott,[4] and Lewis.[5] The formulation of the theory as done by Molière,[3] and Goudsmit and Saunderson[2] has the very interesting feature that the differential law of scattering enters into the theory of multiple scattering only through a single parameter, the screening parameter χ_α. Bethe[6] has established that the theory of Goudsmit and Saunderson[2] has a close quantitative relation to that of Molière.[3] The theory of Molière has been widely applied in the interpretation of experimental results. However, Nigam, Sundaresan, and Wu[7] have pointed out that the formula given by Molière for the scattering cross section of a charged particle by an atom in his theory of multiple scattering is inconsistent. This is because Molière's calculation of the scattering amplitude includes an inconsistent expansion of the phase shift in powers of $\alpha_1=zZe^2/\hbar v$. Nigam *et al.*,[7] use Dalitz's[8] relativistic expression for the single scattering cross section derived in the second Born approximation for the scattering of a spin-half-charged particle by the screened Coulomb field of an atom, and the distribution function for multiple scattering was calculated in powers of α_1 in a consistent manner. They obtained satisfactory agreement with the experimental results of Hanson, Lanzl, Lyman, and Scott[9] for the $1/e$ widths of the distribution function for the scattering of 15.6 MeV electrons by Au and Be. Further the work of Nigam, Sundaresan and Wu,[7] (hereafter to be referred as paper A), in contrast to Molière's[3] theory, predicts different screening angles for electron and positron scattering and consequently, different distribution functions for multiple scattering. Nigam and Mathur[10] have applied the results of paper A and calculated the difference in multiple scattering of electron and positron and found good agreement with the experiment of Henderson and Scott.[11]

The method of estimating the energy of fast ionizing particles in photographic emulsion by measuring the deviations in their tracks produced by multiple scattering was first suggested by Bose and Choudhuri.[12] Gottstein, Menon, Mulvey, O'Ceallaigh, and Rochat[13] have shown that the mean deviation of a charged particle passing through a given layer of matter is directly proportional to the charge and inversely proportional to the product (momentum $\times$ velocity) the constant of proportionality depending on the composition of the scattering medium. They calculated the "scattering constant" using Molière's theory. In this paper, the mean angle of multiple scattering, spatial and pro-

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jected, is calculated (Secs. IV and V) using the distribution function derived in paper A.[7]

II. SUMMARY AND RESULTS OF PAPER A

Molière's theory[3] of multiple scattering is characterized by the definition of a parameter B which depends on the screening angle χ_α. The screening angle χ_α (for multiple scattering) is defined such that it represents all the small angle contributions arising in the angular distribution function for multiple scattering; the cross section for single scattering being used explicitly in the distribution function. For the scattering of a charged particle by the screened Coulomb field of an atom, the screening angle is given by

$$\ln(2/\chi_\alpha)-\tfrac{1}{2}=\int_0^1 [q(y)/y]dy, \quad y=\sin(\chi/2), \tag{1}$$

$$q(y)=\sigma(y)/\sigma_R(y), \tag{2}$$

where χ is the angle of scattering and $q(y)$ is the ratio of the scattering cross section (with screening) to the Rutherford cross section (no screening) for single scattering. The above definition of the screening angle χ_α is such that in the first Born approximation, when $q(y)\rightarrow q_B(y)=\sigma_B(y)/\sigma_R(y)$, for the scattering of an electron of momentum p by an exponentially screened potential $V(r)=-(Ze^2/r)e^{-\lambda r}$, we have

$$\chi_\alpha\rightarrow\chi_0=\hbar\lambda/p. \tag{3}$$

The screened Coulomb potential used by Molière[3] was $V(r)=-(Ze^2/r)\omega(r\lambda_0)$, where the Thomas-Fermi function $\omega(r\lambda_0)$ consisted of a sum of three exponentials and $\lambda_0=Z^{1/3}/0.855a_0$, a_0 being the Bohr radius. In the first Born approximation, this Thomas-Fermi field gives, upon numerical integration,

$$\chi_\alpha\rightarrow\chi_0\simeq 1.13(\hbar\lambda_0/p). \tag{4}$$

With this choice of the potential, Molière[3] calculated the formula for single scattering which then formed the basis of his theory of multiple scattering. It has been, however, pointed out by Nigam, Sundaresan, and Wu[7] (Sec. V, reference 7) that Molière's[3] single scattering formula is based on an inconsistent expansion of the phase shift in powers of $\alpha_1=zZe^2/\hbar v$.

In the paper of Nigam *et al.*,[7] the angular distribution function for multiple scattering is derived by using for single scattering Dalitz's[8] relativistic formula, derived in the second Born approximation, for the scattering of a spin-half-particle of charge z ($z=-1$ for an electron) by an exponentially screened Coulomb potential:

$$V(r)=(zZe^2/r)e^{-\lambda r}, \tag{5}$$

where the screening parameter $\lambda=\mu\lambda_0$, μ being an adjustable parameter of the order of unity. The parameter μ is introduced in order to compensate for the use of a single exponential as the screening factor of the Coulomb field of an atom instead of a sum of three exponentials as done by Molière.[3] The multiple-scattering distribution function was derived by following the simplified procedure of Bethe.[6] According to Goudsmit and Saunderson,[2] the angular distribution is given by

$$f(\theta,t)=\sum_{l=0}^{\infty}(l+\tfrac{1}{2})P_l(\cos\theta) \times\exp\left\{-Nt\int_0^\pi d\chi\sin\chi\sigma_D(\chi)[1-P_l(\cos\chi)]\right\}, \tag{6}$$

where $f(\theta,t)\sin\theta d\theta$ is the actual number of scattered particles between θ and $\theta+d\theta$, t the foil thickness, and N is the number of scattering atoms/cm^3. The expression for the screening angle χ_α is obtained by calculating the integral in the exponential of Eq. (6) and combining all the contribution from the small angles into a single term. This results in

$$\chi_\alpha=\chi_0\left\{1+2\alpha\chi_0\left[\frac{1-\beta^2}{\beta}\ln\chi_0+\frac{0.2310}{\beta}+1.448\beta\right]\right\}, \tag{7}$$

where

$$\chi_0=\mu\frac{\hbar}{p}\left(\frac{Z^{1/3}}{0.885a_0}\right), \quad \alpha=-zZ/137, \quad \beta=v/c. \tag{8}$$

The expression of the screening angle as obtained by Molière is

$$\chi_\alpha=\chi_0\{1.13+3.76\alpha^2/\beta^2\}^{1/2}. \tag{9}$$

The coefficient 1.13 arises from the use of the sum of three exponentials to represent the Thomas-Fermi potential. However, a comparison of Eqs. (7) and (9) shows that Molière's[3] correction term $3.76\alpha^2/\beta^2$ is large for high Z. Thus, the effect of the deviation from the first Born approximation on the screening angle, as estimated by Nigam *et al.*,[7] is much smaller than as given by Molière. This is as it should be since the deviation from the first Born approximation is very small at small angles which go into the definition of the screening angle. It should also be noted that Eq. (7) predicts different screening angle, and hence, different multiple scattering[10,11] for electrons ($z=-1$) and positron ($z=+1$) scattering, in contrast to Molière's[3] theory in which only even powers of α appear. The parameters in terms of which multiple scattering is described are χ_α', ξ, b and B defined as follows:

$$\begin{aligned}\ln(2/\chi_\alpha)&=\ln(2/\chi_\alpha')-\tfrac{1}{2}+C-(2\alpha\chi_0/\beta)(1-\beta^2)(1-C),\\ \xi&=1+(2\alpha\chi_0/\beta)(1-\beta^2),\\ b&=\xi\ln(\chi_c^2/4)-\ln(\chi_\alpha'^2/4),\\ b&=B-\xi\ln B,\end{aligned} \tag{10}$$

where $C=0.577\,215$ is the Euler's constant. Finally, the

TABLE I. Numerical values of the distribution function.[a]

$\vartheta=\theta/\chi_c\sqrt{B}$	$f^{(1)}(\vartheta)$	$\frac{1}{2}f^{(2)}(\vartheta)$	$f^{(1)\prime}(\vartheta)/(-\pi\alpha\beta\chi_c\sqrt{B})$	$f^{(2)\prime}(\vartheta)/(-2\pi\alpha\beta\chi_c(B)^{\frac{1}{2}}\xi)$
0	0.8456	2.4929	3.5449	3.7389
0.2	0.7038	2.0694	3.3374	3.1776
0.4	0.3437	1.0488	2.7741	1.7700
0.6	−0.0777	−0.0044	2.0067	0.1707
0.8	−0.3981	−0.6068	1.2176	−0.9791
1.0	−0.5285	−0.6359	0.5545	−1.3786
1.2	−0.4770	−0.3086	0.09288	−1.1335
1.4	−0.3180	0.0525	−0.1654	−0.5830
1.6	−0.1396	0.2423	−0.2659	−0.05923
1.8	−0.0006	0.2386	−0.2696	+0.2601
2.0	+0.0782	0.1316	−0.2285	0.3611
2.2	0.1054	0.0196	−0.1766	0.3198
2.4	0.1008	−0.0467	−0.1305	0.2243
2.6	0.08262	−0.0649	−0.09521	0.1325
2.8	0.06247	−0.0546	−0.07024	0.06713
3.0	0.04550	−0.03568	−0.05302	0.02865
3.2	0.03288	−0.01923		
3.4	0.02402	−0.00847	−0.03263	0.000807
3.6	0.01791	−0.00264		
3.8	0.01366	0.00005	−0.02196	−0.003219
4.0	1.0638×10^{-2}	0.10741×10^{-2}		
4.2			−0.01556	−0.002795
4.5	0.6140×10^{-2}	0.12294×10^{-2}		
4.6			−0.01159	−0.002025
5.0	0.3831×10^{-2}	0.08326×10^{-2}	−0.00884	−0.001493

[a] There are some errors in the values of $f^{(1)\prime}(\vartheta)$ as reported in references 7 and 10. The author is thankful to Gerald Dick of Argonne National Laboratory for help in the computation of the numerical values of $f^{(1)\prime}$ and $f^{(2)\prime}$.

distribution function can be approximated by

$$f(\theta,t)\simeq K(\chi_c^2B)^{-1}\left\{f^{(0)}(\vartheta)+\frac{1}{B}[f^{(1)\prime}(\vartheta)+f^{(1)}(\vartheta)]+\frac{1}{2!B^2}[f^{(2)\prime}(\vartheta)+f^{(2)}(\vartheta)]+\cdots\right\}, \tag{11}$$

where

$$\int_0^\infty du\,u^{1+\frac{1}{2}\chi_c^2[\beta^2+\pi\alpha\beta-\frac{1}{4}\xi]}J_0(\vartheta u)\exp(-u^2/4)\begin{cases} 1 & =f^{(0)}(\vartheta) \\ -\pi\alpha\beta\chi_c(B)^{\frac{1}{2}}u & =f^{(1)\prime}(\vartheta) \\ \xi(u^2/4)\ln(u^2/4) & =f^{(1)}(\vartheta) \\ -2\pi\alpha\beta\chi_c(B)^{\frac{1}{2}}\xi u(u^2/4)\ln(u^2/4) & =f^{(2)\prime}(\vartheta) \\ [\xi(u^2/4)\ln(u^2/4)]^2 & =f^{(2)}(\vartheta) \end{cases} \tag{12}$$

and

$$K=\exp\left\{\frac{B\chi_c^2}{16}\left[1+\frac{8\pi\alpha\beta}{B}+\frac{2\xi\ln2}{B}+\frac{8(\beta^2+\pi\alpha\beta)}{B}[C-\ln(\chi_c\sqrt{B})]\right]\right\},$$
$$\chi_c^2=4\pi Nte^4z^2Z(Z+1)/(pc\beta)^2,\quad \vartheta=\theta/(\chi_c\sqrt{B}). \tag{13}$$

The exponent of u in the integrand can be taken as unity. ξ is also very close to unity (within a percent or two) and its actual value may be of importance when comparing particles of equal mass and energy but of opposite charges. The integrals in Eq. (12), except for $f^{(0)}$, $f^{(1)}$ and $f^{(1)\prime}$, cannot be carried out analytically. However, for large values of ϑ, one can obtain asymptotic expansions which are in general more useful. In Table I, we have listed the numerical values of $f^{(1)}(\vartheta)$ and $f^{(2)}(\vartheta)$, as given by Bethe,[6] and of $f^{(1)\prime}(\vartheta)/(-\pi\alpha\beta\chi_c\sqrt{B})$ and $f^{(2)\prime}(\vartheta)/(-2\pi\alpha\beta\chi_c(B)^{\frac{1}{2}}\xi)$ as calculated by numerical integration at the Argonne computer IBM 704.

III. ASYMPTOTIC EXPRESSIONS

The asymptotic expressions for $f^{(1)}$, $f^{(1)\prime}$, etc., can be obtained by using the integration formulas given in the Appendix of Molière's paper.[3] The following are the results:

$$f^{(0)}(\vartheta)=2e^{-\vartheta^2},\quad \text{(exact)} \tag{14a}$$

$$f^{(1)}(\vartheta)\simeq\frac{2}{\vartheta^4}+\frac{8}{\vartheta^6}+\frac{36}{\vartheta^8}+\frac{192}{\vartheta^{10}}+\cdots, \tag{14b}$$

$$f^{(2)}(\vartheta)\simeq 32\vartheta^{-6}[\ln(\gamma\vartheta)-\tfrac{3}{2}]+288\vartheta^{-8}[\ln(\gamma\vartheta)-(11/6)]+\cdots, \tag{14c}$$

$$\begin{aligned}I_1(\vartheta)&=f^{(1)\prime}(\vartheta)/(-\pi\alpha\beta\chi_c\sqrt{B})\\ &=4\sum_{n=0}^{\infty}(-1)^n\frac{(n+\frac{1}{2})!}{n!(-n-\frac{3}{2})!}\vartheta^{-2(n+\frac{3}{2})}\simeq-\vartheta^{-3}-2.25\vartheta^{-5}-7.03125\vartheta^{-7}-28.711\vartheta^{-9}-145.35\vartheta^{-11}-\cdots,\end{aligned} \tag{14d}$$

$$\begin{aligned}I_2(\vartheta)&=f^{(2)\prime}(\vartheta)/(-2\pi\alpha\beta\chi_c(B)^{\frac{1}{2}}\xi)\\ &=4\sum_{n=0}^{\infty}\frac{(n+\frac{3}{2})!}{n!(-n-\frac{5}{2})!}\vartheta^{-2(n+\frac{5}{2})}[-2\ln\vartheta+\Psi(n+\tfrac{3}{2})+\Psi(-n-\tfrac{5}{2})]\\ &\simeq 4.5\vartheta^{-5}(-\ln\vartheta+0.703157)+28.125\vartheta^{-7}(-\ln\vartheta+1.103157)\\ &\qquad+172.266\vartheta^{-9}(-\ln\vartheta+1.388871)+1162.79\vartheta^{-11}(-\ln\vartheta+1.16111)+\cdots,\end{aligned} \tag{14e}$$

where[14]

$$\begin{gathered}(n+\tfrac{1}{2})!=\sqrt{\pi}\frac{(2n+1)!!}{2^{n+1}},\quad(-n-\tfrac{1}{2})!=\sqrt{\pi}\frac{(-2)^n}{(2n-1)!!},\quad(-\tfrac{1}{2})!=\sqrt{\pi},\\ (2n+1)!!=1\times3\times5\cdots\times(2n+1),\\ \Psi(x)=\frac{d}{dx}\ln(x!)=\frac{d}{dx}\ln\Gamma(1+x),\\ \Psi(n)=-C+\sum_{x=1}^{n}\frac{1}{x},\quad\Psi(-\tfrac{1}{2}\pm n)=-\ln(4\gamma)+2\sum_{x=0}^{n-1}\frac{1}{2x+1},\quad\Psi(-\tfrac{1}{2})=-\ln(4\gamma),\\ C=0.577215=\ln\gamma.\end{gathered} \tag{15}$$

IV. MEAN SPATIAL ANGLE OF SCATTERING

In order to determine the momentum of a charged particle, as was first suggested by Bose and Choudhuri[12] and later followed up by Gottstein, Menon, Mulvey, O'Ceallaigh, and Rochat,[13] and Fichtel and Friedlander,[15] the important variable to measure along the track of the fast charged particle in nuclear emulsion and cloud-chamber is the mean angle of scattering produced by multiple scattering in traveling through a certain thickness t (cell size). Since $f(\theta,t)\sin\theta d\theta$ is the number of particles scattered in the angular interval $d\theta$, the mean angle of scattering is given by

$$\bar{\theta}=\int_0^{\pi}\theta f(\theta,t)\sin\theta d\theta, \tag{16}$$

so that

$$\bar{\vartheta}=\bar{\theta}/(\chi_c\sqrt{B})=\int_0^{\pi/\chi_c\sqrt{B}}\vartheta\frac{\sin(\vartheta\chi_c\sqrt{B})}{\chi_c\sqrt{B}}f(\vartheta,t)d\vartheta, \tag{17}$$

where

$$f(\theta,t)\sin\theta d\theta=f(\vartheta,t)\frac{\sin(\vartheta_c\chi\sqrt{B})}{\chi_c\sqrt{B}}d\vartheta, \tag{18}$$

with

$$f(\vartheta,t)=K\left\{f^{(0)}(\vartheta)+\frac{1}{B}[f^{(1)\prime}(\vartheta)+f^{(1)}(\vartheta)]+\frac{1}{2!B^2}[f^{(2)\prime}(\vartheta)+f^{(2)}(\vartheta)]+\cdots\right\}. \tag{19}$$

The results for the mean angle of scattering given by Gottstein *et al.*[13] is from Molière.[3] In Molière's calculations, as also in the derivation of Molière's results by Bethe,[6] the sine of the angle is replaced by the angle itself and the upper limit π of the angular integration is replaced by infinity. It is quite easy to see that if this procedure of approximation is now followed, it would lead to a logarithmically divergent term in the contribution to $\bar{\vartheta}$ arising from $f^{(1)\prime}$. If we try to interpret Molière's approximation, then it is clear from Eqs. (16) and (17) that Molière[3] has calculated $\bar{\vartheta}$ in the limit of $\chi_c\sqrt{B}\rightarrow 0$. Since $\chi_c\sqrt{B}$ is generally a small number this approximation in Molière's[3] calculations is quite good. However, since both $f^{(1)\prime}$ and $f^{(2)\prime}$ are proportional to $\chi_c\sqrt{B}$ and the integrations involved in the calculation of the mean angle give additional factors of the type $\ln(\pi/\chi_c\sqrt{B})$, the substitution of $\chi_c\sqrt{B}\rightarrow 0$ from the very beginning is not desirable if we want to

[14] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, Inc., New York, 1945).

[15] C. Fichtel and M. W. Friedlander, Nuovo Cimento **10**, 1032 (1958).

estimate the effect of $f^{(1)\prime}$ and $f^{(2)\prime}$ terms to the magnitude of $\bar{\vartheta}$.

In order to carry out the integration in Eq. (17), the interval of integration was broken into two intervals 0 to ϑ_0 and ϑ_0 to $\pi/\chi_c\sqrt{B}$ such that in the range 0 to ϑ_0, $\sin(\vartheta\chi_c\sqrt{B})\rightarrow\vartheta\chi_c\sqrt{B}$ and numerical values of $f(\vartheta,t)$ were used. In the range ϑ_0 to $\pi/\chi_c\sqrt{B}$, the asymptotic expressions, Eqs. (14a)–(14e), were used. All integrals are then expressible in terms of

$$\int_{\vartheta_0\chi_c\sqrt{B}}^{\pi} dx\frac{\sin x}{x} \quad\text{and}\quad \int_{\vartheta_0\chi_c\sqrt{B}}^{\pi} dx\frac{\cos x}{x},$$

which were calculated analytically to less than 0.1% accuracy. In view of the small value of $\chi_c\sqrt{B}$ and the accuracy of the asymptotic expressions, Eqs. (14), the choice of $\vartheta_0=5$ was considered appropriate. In the following we give the first few significant terms of the results of integration.

$$\bar{\vartheta}_0=\int_0^\infty-\int_{\pi/\chi_c\sqrt{B}}^\infty\left[\vartheta\frac{\sin(\vartheta\chi_c\sqrt{B})}{\chi_c\sqrt{B}}2e^{-\vartheta^2}d\vartheta\right] \simeq\frac{\sqrt{\pi}}{2}e^{-\chi_c^2B/4}, \tag{20a}$$

$$\bar{\vartheta}_1=0.8699-1.5339\chi_c\sqrt{B}+2.3820(\chi_c\sqrt{B})^2, \tag{20b}$$

$$\bar{\vartheta}_2=-0.2088+4.2412\chi_c^2B, \tag{20c}$$

$$\bar{\vartheta}_{1'}=-\pi\alpha\beta\chi_c\sqrt{B}\left[0.9738-\ln(\pi/\chi_c\sqrt{B}) +\chi_c^2B\left(-6.7968+0.375\ln\frac{\pi}{\chi_c\sqrt{B}}\right)\right], \tag{20d}$$

$$\bar{\vartheta}_{2'}=-2\pi\alpha\beta\xi\chi_c\sqrt{B} \times\left[0.2922+\chi_c^2B\left\{\left(\ln\frac{\pi}{\chi_c\sqrt{B}}\right)^2(2.25-0.1172\chi_c^2B) +\left(\ln\frac{\pi}{\chi_c\sqrt{B}}\right)(-7.7070+0.3772\chi_c^2B) +(6.934+0.3036\chi_c^2B)\right\}\right], \tag{20e}$$

where

$$\bar{\vartheta}_n=\int_0^{\pi/\chi_c\sqrt{B}}\vartheta\frac{\sin(\vartheta\chi_c\sqrt{B})}{\chi_c\sqrt{B}}f^{(n)}(\vartheta,t)d\vartheta =\int_0^{\vartheta_0=5}\vartheta^2f^{(n)}(\vartheta,t)d\vartheta+\int_{\vartheta_0}^{\pi/\chi_c\sqrt{B}}\vartheta\frac{\sin(\vartheta\chi_c\sqrt{B})}{\chi_c\sqrt{B}} \times f^n(\vartheta,t)d\vartheta, \tag{21}$$

n taking the values 0, 1, 1′, 2, 2′, etc. In $\bar{\vartheta}_0$ the contribution

$$\int_{\pi/\chi_c\sqrt{B}}^\infty\vartheta\frac{\sin(\vartheta\chi_c\sqrt{B})}{\chi_c\sqrt{B}}\cdot[2e^{-\vartheta^2}d\vartheta]$$

is neglected since $\pi/\chi_c\sqrt{B}\gtrsim 30$–50, so that the factor $e^{-\vartheta^2}$ in the integrand would lead to a vanishingly small value for the integral. Substituting Eqs. (20) and Eq. (17) we obtain the following expression for the mean angle of scattering

$$\bar{\vartheta}=K\left[\bar{\vartheta}_0+\frac{1}{B}(\bar{\vartheta}_{1'}+\bar{\vartheta}_1)+\frac{1}{2!B^2}(\bar{\vartheta}_{2'}+\bar{\vartheta}_2)+\cdots\right] \tag{22}$$

where $\bar{\vartheta}_n$ are given by Eqs. (20) and K and χ_c are defined by Eq. (13). In the limit of $\chi_c\sqrt{B}\rightarrow 0$, Eq. (22) goes over to the result due to Molière,[3] and Gottstein *et al.*,[13] viz.,

$$[\bar{\vartheta}]_{\chi_c\sqrt{B}\rightarrow 0}=\frac{\sqrt{\pi}}{2}\left[1+\frac{0.982}{B}-\frac{0.117}{B^2}+\cdots\right]. \tag{23}$$

V. MEAN PROJECTED ANGLE OF SCATTERING

In cloud chamber experiments and emulsion work one does not measure the spatial angle of scattering θ. Instead one determines its projection on a plane of observation which contains the direction of incidence. Molière[3] denotes this angle by ϕ. Then following Molière, the angular distribution function in terms of $\varphi=\phi/\chi_c\sqrt{B}$ is given by

$$f(\varphi)d\varphi=K\left\{f^{(0)}(\varphi)+\frac{1}{B}[f^{(1)\prime}(\varphi)+f^{(1)}(\varphi)]+\frac{1}{2!B^2}[f^{(2)\prime}(\varphi)+f^{(2)}(\varphi)]+\cdots\right\}d\varphi, \tag{24}$$

where

$$f^{(n)}(\varphi)\simeq\left(\frac{2}{\pi}\right)\int_0^\infty dy\cos(y\varphi)e^{-y^2/4}\left(\frac{y^2}{4}\ln\frac{y^2}{4}\right)^n;\quad f^{(0)}(\varphi)=\frac{2}{\sqrt{\pi}}e^{-\varphi^2},$$

$$f^{(1)\prime}(\varphi)\simeq\left(\frac{2}{\pi}\right)(-\pi\alpha\beta\chi_c\sqrt{B})\int_0^\infty dy\,y\cos(y\varphi)e^{-y^2/4}, \tag{25}$$

$$f^{(2)\prime}(\varphi)\simeq\left(\frac{2}{\pi}\right)(-2\pi\alpha\beta\chi_c\sqrt{B})\int_0^\infty dy\,y\cos(y\varphi)e^{-y^2/4}\left(\frac{y^2}{4}\ln\frac{y^2}{4}\right).$$

The asymptotic expressions for $f^{(1)}(\varphi)$ and $f^{(2)}(\varphi)$ are given by Molière[3] who has also tabulated their numerical values from $\varphi=0$ to 4. For completeness sake we reproduce these values in Table II. For large $\varphi(\varphi>4)$,

$$f^{(1)}(\varphi)\simeq\varphi^{-3}+3\varphi^{-5}+11.25\varphi^{-7}+52.5\varphi^{-9}+\cdots \tag{26a}$$

$$f^{(2)}(\varphi)\simeq6\varphi^{-5}[\ln(2\gamma\varphi)-(25/12)]+45\varphi^{-7}[\ln(2\gamma\varphi)-(49/20)]+\cdots. \tag{26b}$$

The corresponding expressions for $f^{(1)\prime}(\varphi)$ and $f^{(2)\prime}(\varphi)$ can be obtained from Eqs. (A4) and (A6) and are as follows:

$$f^{(1)\prime}(\varphi)\simeq2\alpha\beta\chi_c\sqrt{B}[\varphi^{-2}+1.5\varphi^{-4}+3.75\varphi^{-6}+13.125\varphi^{-8}+59.063\varphi^{-10}+\cdots] \tag{26c}$$

$$f^{(2)\prime}(\varphi)\simeq4\alpha\beta\chi_c\sqrt{B}\left\{3\varphi^{-4}\left[-\ln(2\gamma\varphi)+\frac{11}{6}\right]+15\varphi^{-6}\left[-\ln(2\gamma\varphi)+\frac{137}{60}\right]+\cdots\right\}. \tag{26d}$$

The integral occurring in $f^{(1)\prime}(\varphi)$ can be transformed[16] to (see Appendix)

$$\int_0^\infty dyye^{-y^2/4}\cos(y\varphi)=2-4\varphi e^{-\varphi^2}\int_0^\varphi dte^{t^2}. \tag{27}$$

The numerical values of $f^{(1)\prime}(\varphi)/(-2\alpha\beta\chi_c\sqrt{B})$ are given in Table II.

We can now calculate the mean projected angles of scattering. It is defined by the integral

$$\bar{\varphi}=\int_0^{\delta=\pi/\chi_c\sqrt{B}}d\varphi\varphi f(\varphi) \tag{28}$$

$$=K\left\{\bar{\varphi}_0+\frac{1}{B}[\bar{\varphi}_{1'}+\bar{\varphi}_1]+\frac{1}{2!B^2}[\bar{\varphi}_{2'}+\bar{\varphi}_2]+\cdots\right\}, \tag{28}$$

TABLE II. Numerical values of the distribution function.

$\varphi=\phi/\chi_c\sqrt{B}$	$f^{(1)}(\varphi)$	$\frac{1}{2}f^{(2)}(\varphi)$	$f^{(1)\prime}(\varphi)/(-2\alpha\beta\chi_c\sqrt{B})$
0	+0.0206	+0.416	2.0
0.2	−0.0246	0.299	1.84420
0.4	−0.1336	0.019	1.42409
0.6	−0.2440	−0.229	0.86057
0.8	−0.2953	−0.292	0.29728
1.0	−0.2630	−0.174	−0.15232
1.2	−0.1622	+0.010	−0.43492
1.4	−0.0423	0.138	−0.55643
1.6	+0.0609	0.146	−0.55961
1.8	0.1274	0.094	−0.49677
2.0	0.147	0.045	−0.41072
2.2	0.142	−0.049	−0.32770
2.4	0.1225	−0.071	−0.25901
2.6	0.100	−0.064	−0.20652
2.8	0.078	−0.043	−0.16776
3.0	0.059	−0.024	−0.13926
3.2	0.045	−0.010	−0.11791
3.4			−0.10153
3.5	0.0316	+0.001	
3.6			−0.088605
3.8			−0.078159
4.0	0.0194	0.006	−0.069568

[16] *Tables of Integral Transforms*, Bateman Manuscript Project (McGraw-Hill Book Company, Inc., New York, 1954), Vol. I, p. 158.

where again we have avoided taking the upper limit to be infinite as done by Molière[3] and

$$\bar{\varphi}_0=\frac{1}{\sqrt{\pi}}(1-e^{-\delta^2}),\quad \delta=\pi/(\chi_c\sqrt{B}), \tag{29a}$$

$$\bar{\varphi}_1\simeq0.5538-\delta^{-1}-\delta^{-3}, \tag{29b}$$

$$\bar{\varphi}_2\simeq-0.1320-\tfrac{1}{3}\delta^{-3}[\ln(2\gamma\delta)+\tfrac{1}{3}], \tag{29c}$$

$$\bar{\varphi}_{1'}\simeq(-2\alpha\beta\chi_c\sqrt{B})[0.01804-\ln\delta+0.75\delta^{-2}], \tag{29d}$$

$$\bar{\varphi}_{2'}\simeq(-4\alpha\beta\chi_c\sqrt{B})[0.2886+\delta^{-2}(1.5\ln\delta-0.0945)]. \tag{29e}$$

In the calculation of $\bar{\varphi}_0$ and $\bar{\varphi}_{2'}$, it was found convenient to carry out first the φ integration and then, using Eqs. (A4), (A5), (A6), and (A8), carry out the y integration. $\bar{\varphi}_1$, $\bar{\varphi}_2$ and $\bar{\varphi}_{1'}$ were obtained by integrating numerically over φ from $\varphi=0$ to 4, using the numerical values of $f^{(1)}(\varphi)$, $f^{(2)}(\varphi)$ and $f^{(1)\prime}(\varphi)$ given in Table II, and from $\varphi=4$ to $\varphi=\delta$, the integration over φ was carried out analytically. In the limit of $\chi_c\sqrt{B}\to0$, we obtain from Eqs. (28) and (29), the following result

$$[\bar{\varphi}]_{\chi_c\sqrt{B}\to0}=(1/\sqrt{\pi})[1+(0.982/B)-(0.117/B^2)+\cdots] \tag{30}$$

which is in agreement with Molière[3] and Gottstein *et al.*[13]

It should be noted that the leading terms in Eqs. (20d) and (29d) are proportional to $\ln(\pi/\chi_c\sqrt{B})$. Further the ratio of the leading terms in $\bar{\vartheta}_n$ and $\bar{\varphi}_n$ is $\simeq\pi/2$, as is to be expected from Eqs. (23) and (30).

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APPENDIX

In the following we list the various integrals used in the text ($p\neq$integer). The results (A1) to (A5) are from Molière.[3]

$$\int_0^\infty ydyJ_0(xy)(y^2/4)^p=\frac{2p!}{(-p-1)!}x^{-2p-2},\quad \text{for } p>-1. \tag{A1}$$

Differentiating this once and twice with respect to p, we obtain

$$\int_0^\infty ydyJ_0(xy)(y^2/4)^p \ln(y^2/4)$$
$$=\frac{2p!}{(-p-1)!}[-2\ln x+\Psi(p)+\Psi(-p-1)]x^{-2p-2}, \quad \text{(A2)}$$

and

$$\int_0^\infty ydyJ_0(xy)(y^2/4)^p\left(\ln\frac{y^2}{4}\right)^2$$
$$=\frac{2p!}{(-p-1)!}\{[-2\ln x+\Psi(p)+\Psi(-p-1)]^2$$
$$+\Psi'(p)-\Psi'(-p-1)\}x^{-2p-2}, \quad \text{(A3)}$$

where

$$\Psi(x)=\frac{d}{dx}\ln(x!).$$

$$\int_0^\infty dye^{-\alpha y^2/4}\cos(xy)=\sqrt{\frac{\pi}{\alpha}}e^{-x^2/\alpha}. \quad \text{(A4)}$$

$$\int_0^\infty dy(y^2/4)^p\cos(xy)=\sqrt{\pi}\frac{(p-\frac{1}{2})!}{(-p-1)!}x^{-2p-1},$$
$$\text{for} \quad p>-\tfrac{1}{2}. \quad \text{(A5)}$$

Differentiating this with respect to p we obtain

$$\int_0^\infty dy\cos(xy)(y^2/4)^p\ln(y^2/4)$$
$$=\sqrt{\pi}\frac{(p-\frac{1}{2})!}{(-p-1)!}x^{-2p-1}$$
$$\times[-2\ln x+\Psi(p-\tfrac{1}{2})+\Psi(-p-1)]. \quad \text{(A6)}$$

$$\int_0^\infty dyy(y^2/4)^m e^{-y^2/4}=2(m!). \quad \text{(A7)}$$

Differentiating this with respect to m we obtain

$$\int_0^\infty dyy\left(\frac{y^2}{4}\right)^m e^{-y^2/4}\ln(y^2/4)=2(m!)\Psi(m). \quad \text{(A8)}$$

The integral[16]

$$\int_0^\infty dte^{-qt}\cos(2\alpha^{1/2}t^{1/2})$$
$$=q^{-1}+i\pi^{1/2}\alpha^{1/2}q^{-3/2}e^{-\alpha/q}\,\mathrm{erf}(i\alpha^{1/2}q^{-1/2}),$$
$$\text{for} \quad \mathrm{Re}q>0 \quad \text{(A9)}$$

where

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x dte^{-t^2}. \quad \text{(A10)}$$

The integral $\int_0^x dte^{t^2}$ has been computed by Terrill and Sweeny[17] for $x=0$ to $x=4$.

[17] H. M. Terrill and L. Sweeny, Jr., J. Franklin Inst. **237**, 495 (1944); **238**, 220 (1944).